IV.

ON THE

FORMULÆ FOR CALCULATING AZIMUTH IN TRIGONOMETRICAL OPERATIONS.

By Captain G. EVEREST, F.R.S. M.A.S., &c.

Surveyor General and Superintendent of the Great Trigonometrical Survey of India.

In offering to the notice of the Society the accompanying paper, I beg to explain that my object is to put on record certain formulæ, connected with the method generally employed in the Trigonometrical Surveys of England and of India, for determining Azimuths.

Those of my readers, who are familiar with this subject, will remember that the method in question consists in observing the difference of Azimuth between a fixed lamp of reference and some circumpolar star, generally α ursæ minoris, at the time of its greatest distance on the east or west side of the meridian.

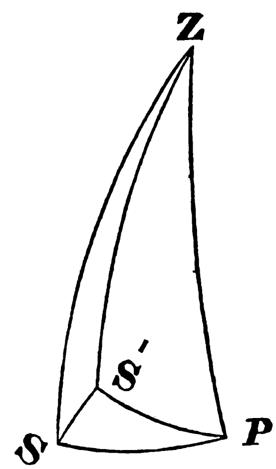
But to accomplish this, the actual time of the phenomenon, and frequently the altitude, require to be known, and as it is advisable to have these elements prepared for the occasion at leisure, the latitude of the place is sometimes drawn from data to which the final corrections have not been applied, and the polar distance is perhaps taken from a catalogue which succeeding observations have shown to be imperfect.

The second part of this paper is intended therefore to furnish formulæ, whereby the observer may introduce the required corrections without undergoing the toil and loss of time which would attend a formal recomputation of the whole set of observations, and it will be remarked that the second hypothesis will also enable him to compute, by means of differences, a series for many nights in succession with quite as much correctness as if entire quantities had been used; for in that case he has only to calculate the elements for the first night, and substitute for the value of d b the increment or decrement of polar distance, the other terms being virtually constant.

It has always, however, been an evil complained of in operations of this kind, that by limiting the case to the actual time of maximum Azimuth, the powers of the observer are much curtailed, because he cannot take more than one observation on the same night.

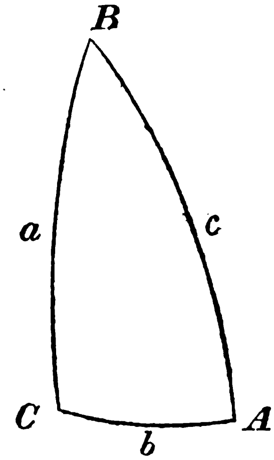
If observations, taken intermediately between the culmination and the time of the greatest Azimuth, were to be computed with reference to the meridian, it would be indispensable to employ large quantities, and the operose formulæ of Spherical Trigonometry, which would not only be laborious, but would not give so much accuracy as the method of eliciting the correction by means of differential terms.

I shall explain this better by a reference to the diagram in the margin, wherein the two Arcs ZS, ZS' are drawn very near to each other, the angle PSZ being a right angle, and it will be seen immediately how much more easily and accurately the angle PZS' may be found by computing the partial angle SZS', and deducting it from PZS, than by direct computation of the entire angle PZS' itself.



I have introduced, in the first part, the ordinary rules for computing the elements at the time of maximum, with the view that those, for whose use these formulæ are intended, may not need a reference to first principles, but have the subject in a complete state before them; and this must be my apology, should it be objected to me that I have presumed to intrude on the Society with propositions strictly elementary.

In any spherical triangle *ABC* if the sides *b* and *c* are constant, the sines of the angles *B* and *C* will attain their greatest values contemporaneously.



For the general equation is

$$sin\ B = sin\ C.\ \frac{sin\ b}{sin\ c}$$

in which the term *sin B* is obviously a maximum when *sin C* is a maximum; *i. e.* when $C = \frac{\pi}{2}$ or 90°.

If, therefore, *A* represent the Pole, *B* the Zenith, and *C* the place of a circumpolar star; when the Azimuth which is represented by the angle *B* is a maximum, the angle of position at *C* will be a right angle: in that case, therefore, we have

1st. $Cos\ A = tan\ b.\ cot\ c$ .... (α)

2nd. $sin\ B = \frac{sin\ b}{sin\ c}$ .... (β)

3rd. $Cos\ a = \frac{cos\ c}{cos\ b}$ .... (γ)

But *A* represents the hour angle, or portion of sidereal space passed through between the instant of transit and that of maximum Azimuth;

c, the complement of the Latitude of the place of observation, or $c = \frac{\pi}{2} - \lambda$; b, the polar distance of the star; B the angle of Azimuth, and a the Zenith distance: consequently, if t^s denote the seconds of sidereal time, the above equations are transformed to

1st. $\cos(15.t^s) = \tan$ ∗'s Pol. dist $\times \tan\lambda$
2nd. $\sin$ ∗'s Azimuth $= \sin$ ∗'s Pol. dist $\times \sec\lambda$
3rd. $\sin$ ∗'s Altitude $= \sec$ ∗'s Pol. dist $\times \sin\lambda$

If t^s be required in mean solar time, it must be diminished in the ratio of twenty-four hours to 23ʜ 56ᴍ 4ˢ, or if τ^s denote the seconds in mean solar time, then we have Log. τ^s = Log. t^s + Const. $\bar{1}$·9988127.

The corrections are also easily obtained from tables constructed on purpose.

To take an example of this; let the time of the greatest Western Azimuth of the Pole-star be required on the 4th May, 1830, in Latitude 24° 0′ 0″, as also the Azimuth and Altitude at that instant; the Longitude being 78° East of Greenwich, and the Polar distance 1° 36′ 0″.

		Hour Angle.	Azimuth.	Altitude.
Pol. dist	1° 36′ 0″	*tan* 8,4461103	*sin* 8.4459409	*sec* 0.0001694
Lat.	24° 0′ 0″	*tan* 9,6485831	*sec* 0.0392698	*sin* 9.6093133
Space	89° 17′ 14.″8	*cos* 8.0946934	*sin* 8.4852107	*sin* 9.6094827
	4		1° 45′ 5.″37	24° 0′ 35.″83.
	5 57 8.98	Siderial time.	Refraction +	0 2 9.83.
	22 16 3.19	*A.R* (∗ — ⊙) at noon.	Appt. Altde	24 2 45.66.
	16 18 54.21	Sid. time		
	— 2 40.37	Corrⁿ.		
	16 16 13.84	Mean Solar Time.		

Hitherto the two sides b and c have been supposed to be correctly known, but it is not an unusual occurrence that a series of observations is computed for many nights by anticipation, with a Latitude merely approximate, and that when this element comes to be finally determined, corrections must be applied to obviate the effects of errors which may thus have been introduced.

To find these corrections, we must differentiate the equations (α), (β), (γ), with respect to c; whence we obtain

1st. $-\sin A.\, dA = -\dfrac{\tan b}{\sin^2 c}.\, dc = \dfrac{\tan b}{\sin^2 c}.\, d\lambda$

$\therefore -dA = \dfrac{\tan b}{\sin A.\, \sin^2 c}.\, d\lambda = \dfrac{\tan b}{\sin a.\, \sin c}.\, d\lambda$

Whence $-dA = \dfrac{\tan B}{\sin c}.\, d\lambda = \tan B.\, \sec \lambda.\, d\lambda$

2nd. $\cos B.\, dB = \dfrac{-\sin b.\, \cos c.\, dc}{\sin^2 c.} = -\sin B.\, \cot c.\, dc$

$\therefore dB = \text{tang } B.\, \text{tang } \lambda.\, d\lambda$

3d. $-\sin a.\, da = \dfrac{-\sin c.\, dc}{\cos b}$

$\therefore da = \dfrac{\sin c}{\sin a.\, \cos b}.\, dc = \dfrac{dc}{\sin A.\, \cos b}$

$\therefore -da = \dfrac{d\lambda}{\cos B} = \sec B.\, d\lambda$

To show the application of these formulæ, let it be supposed that the parts of the triangle of greatest Azimuth had been computed previously for some nights in succession, with a Latitude deduced from an approximate series of triangles, and that instead of 24° 0′ 0″, as supposed in the last example, the Latitude was found to be 24° 1′ 45″. Then the operation will stand as follows :

	Hour Angle.	Azimuth.	Altitude.
$d\lambda = 105''$ .... Log	.... 2·02119	.... Log 2.02119	.... Log 2.02119
$B = 1°\ 45'$ ...... Log tan	$\bar{2}$·48541	.... Tan $\bar{2}$.48541	.... Sec 0.00020
$\lambda = 24°\ 0'\ 0''$ ...... Sec	.... 0.03927	.... Tan $\bar{1}$.64858	....
15 A. C. Log	8·82391		
Corrections...... 0·″23 Log	.... $\bar{1}$·36978	1.″43..Log 0·15518	1′ 45.″05 .. Log 2·02139
1st Computn...	H. ′ ″ 16 16 13·84	1 45 5. 27	24 2 45· 66
Correct values	H. M. S. 16 16 13·61	° ′ ″ 1 45 6, 70	24 4 30· 71

It will also sometimes happen in practice, that a series of observations, computed with data drawn from an imperfect catalogue of former years, requires to be corrected in conformity with the superior accuracy obtained by modern observers. In that case we must differentiate the same equations (α) (β) (γ) with respect to the Polar distance b, whence we obtain

$$\text{1st} - \sin A.\,\mathrm{d}A = \frac{\cot c}{\cos^2 b}.\,\mathrm{d}b$$

$$\therefore -\mathrm{d}A = \frac{\cot c}{\sin A.\cos^2 b}.\,\mathrm{d}b = \frac{\cos c}{\sin c.\sin A.\cos^2 b}.\,\mathrm{d}b$$

$$\text{Or} - \mathrm{d}A = \frac{\cos c}{\sin a.\cos^2 b}.\,\mathrm{d}b = \frac{\cos a}{\sin a.\cos b}.\,\mathrm{d}b = \cot a.\sec b.\,\mathrm{d}b$$

$$\text{2d}\ \cos B.\,\mathrm{d}B = \frac{\cos b.\,\mathrm{d}b}{\sin c}$$

$$\therefore \mathrm{d}B = \frac{\cos b}{\cos B.\sin c}.\,\mathrm{d}b = \frac{\mathrm{d}b}{\sin A.\cos\lambda}$$

$$\text{3d} - \sin a.\,\mathrm{d}a = \frac{\cos c.\sin b.\,\mathrm{d}b}{\cos^2 b}$$

$$\therefore -\mathrm{d}a = \frac{\cos c.\sin b}{\sin a.\cos^2 b}.\,\mathrm{d}b = \cot a.\tan b.\,\mathrm{d}b.$$

These formulæ, computed similarly to the former examples, will stand as follow. Suppose that (instead of 1° 36′) the polar distance had been 1° 35′ 34″, but that all the observations had been already computed with the former of these values. Then we shall have d b = — 26″.

d b = — 26″........Log 1.41497........Log 1.41497........Log $\bar{1}$.41497

b = 1° 36′ 0″......Sec 0.00017..............................Tan 2.44611

$\left(\frac{\pi}{2}-a\right)$ = 24° 0′ 36″ .... Tan $\bar{1}$.64878..............................Tan $\bar{1}$.64878

15 A. C. Log............$\bar{2}$.82391

A = ...Cosec. 0.00003

λ = ...Sec 0.03927

Corrections,..............0.77..Log $\bar{1}$.88783 28″.46.. Log 1.45427..0″.32..Log $\bar{1}$.50986

1st. Compd. values,. 16 16 13.84 ...... 1° 45′ 5 .37 .... 24 2.45.66

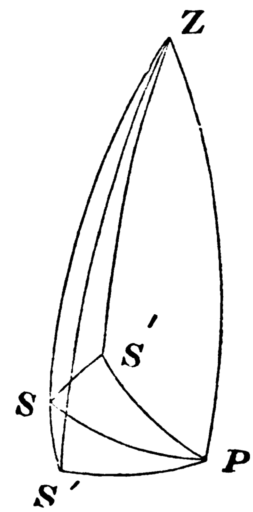
Correct values, ... 16 16 14.61........ 1° 44′ 36.91″ 24 2.45.34

In these computations, the correction is applied to the altitude with an opposite sign to that resulting from the formulæ, as due to the zenith distance, the rationale of which will be evident. The formulæ will evidently shorten such operations considerably, because there is no necessity for more than five places of decimals, unless the variations are very large, and thus, if we retain all the quantities but the variation of b, we may compute a set of observations for many nights in succession, by merely finding the variations which are occasioned in the other parts.

In the work on the great Meridional Arc of India, which the Court of Directors did me the honor to have printed, the principle is examined (vide page 89,) of determining for how long periods some of the principal circumpolar stars of the Greenwich Catalogue may be considered as stationary in Azimuth; and it is therein shewn that, during the 2′ 3.″6 preceding and subsequent to the Maximum, the variation in Azimuth of the Pole Star is only 0.″25, a quantity less than the powers of our best instruments can be considered capable of detecting under ordinary circumstances—similarly β Ursæ Minoris and β Cephei, the two stars in former editions of the Nautical Almanack nearest the Pole, have 1′ 23″ and 1′ 12″ for their stationary periods.

It must, however, be remarked, that the hypothesis, on which that enquiry is conducted, is not rigorous; for it is therein taken for granted that the same vertical circle will pass through the upper and lower positions of the star at equal lapses of time before and after the Maximum, an assumption which, though perfectly admissible for the end therein proposed, will not bear to be much extended: as for instance, suppose it were required to determine what would be the effect on the Azimuth, if instead of the precise instant of the Maximum, the observation were made at any time before or after that phenomenon.

To this end let PSZ be the polar triangle right angled at S, and let S' be the place of a star before or after arriving at S—Draw the Arcs of great Circles PS', SS', ZS', and then since PSS' is Isosceles, if a perpendicular were drawn from P on SS' it would divide that side, and also the angle at P into two equal and similar parts, so that if δP, δZ denote the variations of the hour angle and Azimuth, we have



1st. Tan $PSS' = \text{Cot} \frac{1}{2} \delta P \text{ Sec } PS$

2d. Sin $\frac{1}{2} SS' = \text{Sin} \frac{1}{2} \delta P \text{ Sin } PS$

Hence, because of the right angle at S, we have $\sin ZSS' = \cos PSS'$ and $\cos ZSS' = \pm \sin PSS'$ and therefore the general equation becomes (vide Woodhouse's Trigon. page 157—3d edition).

$$tan\ \delta\ Z = \frac{cos\ PSS'}{cot\ SS'.\ sin\ ZS \mp sin\ PSS'.\ cos\ ZS}$$

But $cos\ ZS = \frac{cos\ PZ}{cos\ PS} = \frac{sin\ \lambda}{cos\ PS}$

And $tang\ ZS = \frac{cos\ Z}{cot\ PZ} = \frac{cos\ Z}{tan\ \lambda}$

$\therefore\ sin\ ZS = \frac{cos\ Z.\ cos\ \lambda}{cos\ PS}$

Consequently $tan\ \delta\ Z = \frac{cos\ PSS'.\ cos\ PS}{cot\ SS'.\ cos\ Z.\ cos\ \lambda \mp sin\ PSS'.\ sin\ \lambda}$

$$= \frac{cot\ PSS'.\ cos\ PS}{sin\ \lambda} . \frac{1}{\frac{cos\ Z}{tan\ SS'.\ tan\ \lambda.\ sin\ PSS'} \mp 1}$$

$$= \frac{cos^2\ PS.\ tan\ \frac{1}{2}\ \delta P}{sin\ \lambda} . \frac{1}{\frac{cos\ Z}{tan\ SS'.\ tan\ \lambda.\ sin\ PSS'} \mp 1}$$

Where the upper sign is used, when the star is nearer the Zenith, and the lower, when it is further removed from the Zenith than in the position of maximum.

If in the former of these cases, which occurs in the Western Elongation before, and in the Eastern after ariving at the maximum; we put

$$\left(\frac{tan\ SS'.\ tan\ \lambda.\ sin\ PSS'.}{cos\ Z}\right)^{\frac{1}{2}} = sin\ \theta$$

Then $tan\ \delta Z = \frac{cos^2\ PS.\ tan\ \frac{1}{2}\ \delta P}{sin\ \lambda} .\ tan^2\ \theta$

Or $\delta\ Z = \frac{cos^2\ PS.\ tan\ \frac{1}{2}\ \delta P.\ tan^2\ \theta}{sin\ \lambda.\ sin\ 1''}$ in seconds of a great circle

Likewise in the latter case, which takes place when before the maximum in the Eastern and after the maximum in the Western Elongation, if we put

$$\left(\frac{\tan SS'. \tan\lambda. \sin PSS'}{\cos Z}\right)^{\frac{1}{2}} = \tan\theta'$$

Then $\tan\delta Z = \frac{\cos^2 PS. \tan\frac{1}{2}\delta P}{\sin\lambda}. \sin^2\theta'$

And $\delta Z = \frac{\cos^2 PS. \tan\frac{1}{2}\delta P. \sin^2\theta'}{\sin\lambda. \sin 1''}$

This method is quite rigorous, but it is rather more operose than the nature of the case usually requires: before, however, proceeding to simplify the formulæ, it may *not* be worth while to give an example of each of the cases above adverted to, and to that end let it be required to determine what would be the correction to be applied to the Azimuth, if instead of the actual instant of maximum on the 4th May 1830, the star had been observed thirty minutes before or after that occurrence.

This computation will be most conveniently arranged according to the following form, premising that, in deducing the tangent of a very small arc from its sine, and vice versâ, the easiest method is to add or subtract the Log. Secant, and that to deduce the tangent of an angle from the sine of half the angle, the easiest way is to add to the latter the Log. of 2, + 3 times Log. Secant; as is evident from the following consideration:

$$\tan 2\theta = \frac{2. \tan\theta}{1 - \tan^2\theta} = 2\tan\theta. (1 + \tan^2\theta + \tan\theta^4, \text{\&c.})$$

$$\therefore \tan 2\theta = 2\sin\theta. \sec\theta. (1 + \tan^2\theta) = 2\sin\theta. \sec^3\theta$$

EXAMPLE.

To find *Tan PSS′* and *Sin PSS′*

$\frac{1}{2}\,\delta P$ (= 15′ in time) = 3° 45′ 0″	L. *Cot*	1.1834706
PS = 1° 36′ 0″	L. *Sec*	0.0001694
PSS'	L. *Tan*	1.1836400

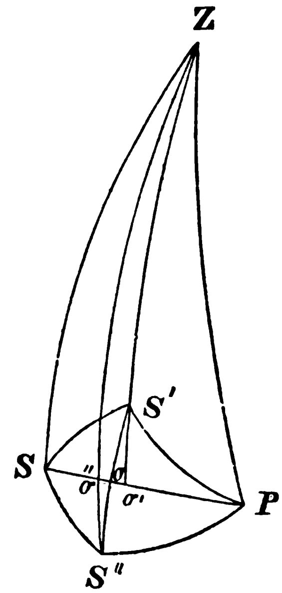
To find *Sin* ½ *SS′* and *Tan SS′*

$\frac{1}{2}\,\delta P$ = 3° 45′ 0″	L. *Sin*	8.8155985
PS = 1° 36′ 0″	L. *Sin*	8.4459409
½ *SS′*	L. *Sin*	7.2615394
3. *Corr*	L. *Sec*	0.0000021
	Log of 2	0.3010300

	L. *Sin*	9.9990698
SS′	L. *Tan*	7.5625715
Sin Z = 8.4852107	L. *Sec*	0.0002028
λ = 24° 0′ 0″	L. *Tan*	9.6485831
	2)	7.2104272

8.6052136	L. *Tan* (θ′)		L. *Sin* (θ)	8.6052136
0.0003528	L. *Sec* (θ′)		L. *Sec* (θ)	0.0003528
8.6048608	L. *Sin* (θ′)		L. *Tan* (θ)	8.6055664
9.9998306	L. *Cos PS*		L. *Cos*	9.9998306
8.6046914				8.6053970
7.2093828	*Double*		*Double*	7.2107940
8.8165294	L. *Tan* ½ δ*P*		L. *Tan*	8.8165294
0.3906867	L. *Cosec* λ		L. *Cosec*	0.3906867
5.3144251	A. C. L. *Sin* 1″		A. C. L. *Sin*	5.3144251
1.7310240	Log. 53″·830		54″·005 Log.	1·7324352
	Value of δ*Z* below.		Value of δ*Z* above.	

It will now be in place to explain how the above formula can be simplified, to which purpose let S', S'' be two positions of a circumpolar star at equal lapses of time before and after the maximum, so that the angles at P, viz. SPS', SPS'' may be equal to each other; then if the diagonal $S'\,S''$ of the Quadrilateral formed by the pole and the three positions of the star be drawn, it will intersect the other diagonal PS at right angles in σ and the two vertical circles ZS', ZS'' (produced in the former case) will intersect the great circle PS obliquely in two points σ', σ'' making two very small triangles $S'\sigma\sigma'$, $S''\sigma\sigma''$, each equal and similar to the other and right angled at σ. Now in $\Delta\, ZS\sigma'$ which is right angled at S, if we denote as before by δZ the variation or $SZ\sigma'$, we have



1st $\sin SZ = \cot \delta Z.\ \tan S\sigma'$; or $\tan S\sigma' = \sin SZ.\ \tan \delta Z$

$\therefore\ S\sigma' = \sin SZ.\ \delta Z = \sin P.\ \cos\lambda.\ \delta Z \ldots\ldots\ldots\ (\alpha)$

2d $\cos\sigma' = \sin\delta Z.\ \cos SZ$

$$\therefore\ \left(\frac{\pi}{2} - \sigma'\right) = \cos SZ.\ \delta Z;\ \text{and}\ \sigma' = \frac{\pi}{2} - \cos SZ.\ \delta Z$$

3d $\therefore\ \sigma S'\sigma'$ or $\angle\ S' = \cos SZ.\ \delta Z$ ———— (in $\Delta\ S'\sigma\sigma'$)

And since $\tan\sigma\sigma' = \sin S'\sigma.\ \tan S'$

Therefore $\sigma\sigma' = S'\sigma.\ \cos SZ.\ \delta Z \ldots\ldots\ldots\ (\beta)$

Again in $\Delta\ PS'\sigma$ right angled at σ we have

1st $\cos\delta P = \cot PS'.\ \tan P\sigma$; but $\tan P\sigma = \tan(PS - S\sigma)$

$$\therefore\ \frac{\tan PS - \tan S\sigma}{1 + \tan PS.\ \tan S\sigma} = \cos\delta P.\ \tan PS$$

$$\therefore\ \tan PS - \frac{S\sigma}{\cos^2 PS} = \cos\delta P.\ \tan PS$$

$\therefore\ S\sigma = tan\ PS.(1 - cos\ \delta P).cos^2 PS = sin\ 2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P. \dots.\ (\gamma)$

2nd $Tan\ S'\sigma = sin\ P\sigma.\ tan\ \delta P = sin\ (PS - S\sigma).\ tan\ \delta P$

$\therefore\ S'\sigma = (sin\ PS - cos\ PS.\ S\sigma).\ tan\ \delta\ P$

Or $S'\sigma = sin\ PS.\ tan\ \delta\ P.\ (1 - cot\ PS.\ S\sigma)$

$= sin\ PS.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P) \dots\dots\dots\dots\dots\dots\ (\epsilon)$

Combining now the equations (β) and (ϵ) we obtain

$\sigma\sigma' = sin\ PS.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P)\ cos\ SZ.\ \delta\ Z$

$= tan\ PS.\ sin\ \lambda.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P).\ \delta\ Z. \dots\dots\dots\ (\zeta)$

$$\left(\text{because } cos\ ZS = \frac{cos\ PZ}{cos\ PS} = \frac{sin\ \lambda}{cos\ PS}\right)$$

and this value of $\sigma\sigma'$ answers for both the upper and lower positions, being subtractive in the former, and additive in the latter, with respect to the mean distance $S\sigma$.

We have, therefore, generally

$S\sigma = S\sigma' \mp \sigma\sigma' \dots\dots$ in which, by substituting the values given in Equations (γ) (α) (ζ) we get

$$Sin\ 2\ PS.\ sin^2 \tfrac{1}{2}\ \delta\ P = (sin\ P.\ cos\ \lambda \mp tan\ PS.\ sin\ \lambda.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \tfrac{1}{2}\ \delta\ P)).\ \delta\ Z$$

$$\therefore\ \delta Z = \frac{sin\ 2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P}{sin\ P.\ cos\lambda.\left\{1 \mp \dfrac{tan\ PS.\ tan\ \lambda.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P)}{sin\ P}\right\}}$$

$$= \frac{sin\ 2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P}{sin\ P.\ cos\ \lambda.\ (1 \mp cot\ P.\ tan\ \delta\ P.\ (1 - 2.\ cos^2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P))}$$

$$\text{Or } \delta\ Z = \frac{sin\ 2\ PS.\ sin^2 \frac{1}{2}\ \delta\ P}{sin\ 1''.\ sin\ P.\ cos\ \lambda}.\ (1 \pm cot\ P.\ tan\ \delta\ P) \text{ nearly.}$$

Whence Log $\delta\ Z$ = Log sin 2 PS + 2. Log. $sin\ \frac{1}{2}\ \delta\ P$ + Log. $cosec\ P$ + Log. $sec\ \lambda$ + A. C. Log. $sin\ 1'' \pm M.\ cot\ P.\ tan\ \delta\ P$. .... Where M denotes the number 0.4342944819, &c. whose Log is 9.6377843, the upper or lower sign being used according as the star is above or below the maximum.

The nature of this last substitution has been shewn in my work on the measurement of an Arc of the Meridian (Page 61) and is simply thus.

$$d(\text{hy. log}(1 \pm x)) = \frac{\pm\, d\, x}{1 \pm x} = \pm\, d\, x.\,(1 \mp x + x^2 \mp x^3 + \&c.)$$

$$\therefore\ \text{hy. log}(1 \pm x) = \pm\,(x \mp \frac{x^2}{2} + \frac{x^3}{3} \mp \frac{x^4}{4} + \&c.)$$

$$\therefore\ \text{Log}\,(1 \pm x) = \pm\, M x.\,(1 \mp \frac{x}{2} + \frac{x^2}{3} \mp \frac{x^3}{4}\ \&c.)$$

in which, when x is very small, the series converge so rapidly that all terms but the first may be omitted, and we get merely

Log. $(1 \pm x) = \pm\, M x.$

Therefore Log. $(1 \pm \cot P.\, \tan \delta P) = \pm\, M.\, \cot P.\, \tan \delta P.$

Taking now the elements as in the first of the above instances, viz.

$PS = 1°\ 36'\ 0''$; $\delta P = 30'$; $P = 89°\ 17'\ 14''.8$; $\lambda = 24°\ 0'\ 0''$

the computation will be as follows:

To find $M. \cot P. \tan \delta P$

9.63778

9.11943 *Tan.* 7° 30′

8.09472 *cot P*

6.85193 Log. of	$\mp$ 0.0007111
$2\,PS = 3°\ 12'\ 0''$ .. Log. *Sin.*	8.7468015
$\frac{1}{2}\,\delta P = 3°\ 45'\ 0''$ .. 2 Log. *Sin.* ..	7.6311970
P Log. *Cosec*	0.0000336
$\lambda = 24°\ 0'\ 0''$ Log. *Sec*	0.0392698
Ar. Co. Log. *Sin* 1″	5.3144251
δZ above 54″. 005	1.7324381
δZ below 53″. 829	1.7310159

The above computations will shew that the approximate method may be quite as much relied on as the more elaborate one, and it will appear on pursuing the enquiry that, for about thirty-two minutes prior and an equal lapse subsequent to the maximum, the Polar Star only varies one minute of space in Azimuth in the latitude of 24° 0′ 0″.